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Optimization of parameters for curve interpolation by cubic splines[☆]

A. Kouibia[*], M. Pasadas, M.L. Rodríguez

Dep. Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

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ABSTRACT

In this paper, we present an interpolation method for curves from a data set by means of the optimization of the parameters of a quadratic functional in a space of parametric cubic spline functions. The existence and the uniqueness of this problem are shown. Moreover, a convergence result of the method is established in order to justify the method presented. The aforementioned functional involves some real non-negative parameters; the optimal parametric curve is obtained by the suitable optimization of these parameters. Finally, we analyze some numerical and graphical examples in order to show the efficiency of our method.

1. Introduction

Different techniques for the construction of a curve have been developed in recent years, for example interpolation by spline functions, based on the minimization of a certain functional in an adequate Sobolev subspace (see [1–5]). Such a functional may represent minimal energy [6], or physical consideration such as minimization of the air of a surface, or minimization of the curvature or variation of curvature (see [7]). These techniques have many applications in CAD, CAGD and Earth Sciences.

In this work, we propose a variational method that consists of minimizing a suitable functional from a Lagrange data set in a parametric space of the cubic spline functions. We study an optimization interpolating method of curves in order to obtain a pleasing shape. By minimizing a quadratic functional that contains some terms associated with Sobolev semi-norms, we obtain a new function that we call the parametric interpolating cubic spline. We study some characterizations of this function, and shall express it as a linear combination of the basis functions of the parametric space of cubic splines. Moreover, under adequate hypotheses, we prove that such a function converges to another given function from which the data proceed.

Sobolev semi-norms are controlled by certain parameters, so here we also study the estimation of the values of these parameters minimizing a given measure, as we establish a method to obtain the optimal parameters that produce the optimal interpolating curve. We finish this work by presenting some numerical and graphical examples in order to show the effectiveness and the validity of our method.

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* Corresponding author. Tel.: +34 958 240478; fax: +34 958 248596.
E-mail addresses: kouibia@ugr.es (A. Kouibia), mpasadas@ugr.es (M. Pasadas), miguelrg@ugr.es (M.L. Rodríguez).

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We can emphasize the relevant difference between this manuscript and others as follows. In [3,7], we present a method of adjustment (smoothing), whereas in this manuscript we outline beforehand a method of interpolation. The approximation techniques applied for a method of interpolation on the one hand, and on the other for adjustment (smoothing), are totally different. Meanwhile, in [2] we used the notion of variational splines in Sobolev space, while in this manuscript we study a discrete problem in a finite dimensional space, namely the space of cubic splines. A further new contribution that we emphasize in this manuscript with regard to the others [3,7,2] is our study of the optimization of the parameters.

The remainder of this paper is organized as follows. In Section 2, we briefly recall some preliminaries and notations. Section 3 is devoted to stating an interpolation problem and to defining and characterizing the parametric interpolating cubic spline. In Section 4, we study how to compute such cubic splines in practice, while a result about convergence is carefully established in Section 5. In Section 6, we present a method for the optimization of parameters. Finally, Section 7 illustrates some numerical and graphical examples.

2. Notations and preliminaries

Let us provide some notations that are necessary to develop this work. For each $k \in \mathbb{N}$, we denote by $\langle \cdot \rangle_k$ and $\langle \cdot, \cdot \rangle_k$, respectively, the Euclidean norm and inner product in $\mathbb{R}^k$. For any $n \in \mathbb{N}^*$ and given $a, b \in \mathbb{R}$ with $a < b$, we consider the interval $R = (a, b)$ and we denote by $\mathbb{P}_n(R)$ the linear space of the real polynomials with a total degree less than or equal to n.

Now, let $H^3(R; \mathbb{R}^k)$ be the usual Sobolev space of (classes of) functions $\mathbf{u}$ belonging to $L^2(R; \mathbb{R}^k)$, together with all their derivatives $\mathbf{u}^{(\ell)}$ with $\ell \in \mathbb{N}$, in the distribution sense, of the order $\ell \leq 3$. This space is equipped with the inner semi-products

$$(\mathbf{u}, \mathbf{v})_\ell = \int_R \left\langle \mathbf{u}^{(\ell)}(x), \mathbf{v}^{(\ell)}(x) \right\rangle_k \mathrm{d}x, \quad 0 \leq \ell \leq 3,$$

the corresponding semi-norms $|\mathbf{u}|_\ell = ((\mathbf{u}, \mathbf{u})_\ell)^{1/2}$, $0 \leq \ell \leq 3$ and the norm $\|\mathbf{u}\| = \left(\sum_{\ell \leq 3} |\mathbf{u}|_\ell^2\right)^{1/2}$.

We briefly write $H^3(R) := H^3(R; \mathbb{R})$ and define the inner semi-products by

$$(\mathbf{u}, \mathbf{v})_{\ell,1} = \int_R \mathbf{u}^{(\ell)}(x)\mathbf{v}^{(\ell)}(x)\mathrm{d}x, \quad 0 \leq \ell \leq 3.$$

We denote by $\|\mathbf{u}\|_0 = \left(\int_R \langle \mathbf{u}(x) \rangle_k^2 \, \mathrm{d}x\right)^{1/2}$ the norm of $L^2(R; \mathbb{R}^k)$.

Let $T_n = \{x_0, \ldots, x_n\}$ be a subset of distinct points of $[a, b]$, with $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. We denote by $S_3(T_n)$ the space of cubic splines given by

$$S_3(T_n) = \left\{ s \in C^2[a, b] \mid s|_{([x_{i-1}, x_i])} \in \mathbb{P}_3([x_{i-1}, x_i]), \ i = 1, \ldots, n \right\}.$$

It is known that $\dim S_3(T_n) = n + 3$. The parametric space of cubic spline functions $V_N = (S_3(T_n))^k$, with $N = k(n + 3) = \dim V_N$, is a Hilbert subspace of $H^3(R; \mathbb{R}^k)$ equipped with the same norm, semi-norms and inner semi-products of $H^3(R; \mathbb{R}^k)$. Moreover, it is verified that

$$V_N \subset H^3(R; \mathbb{R}^k) \cap C^2(\overline{R}; \mathbb{R}^k). \tag{1}$$

Finally, we denote by $\mathcal{M}_{r \times k}$ the space of real matrices with r rows and k columns equipped by the inner product

$$\langle A, B \rangle_{r,k} = \sum_{i,j=1}^{r,k} a_{ij} b_{ij}, \quad \text{with } A = \left(a_{ij}\right)_{\substack{1 \leq i \leq r \\ 1 \leq j \leq k}}, \ B = \left(b_{ij}\right)_{\substack{1 \leq i \leq r \\ 1 \leq j \leq k}},$$

and the corresponding norm $\langle A \rangle_{r,k} = \left(\langle A, A \rangle_{r,k}\right)^{1/2}$.

In this section, the symbol C will denote a positive constant, and does not necessarily denote the same quantity in different formulas.

3. Parametric interpolating cubic spline

Let $\boldsymbol{\Upsilon}_0 \subset \mathbb{R}^k$, with $k = 2, 3$, be a curve considered as an image of the map $\mathbf{f}$, with $\mathbf{f} \in C^2(R; \mathbb{R}^k)$, for the sake of simplicity we say that $\boldsymbol{\Upsilon}_0$ is defined by $\mathbf{f}$. For each $r \in \mathbb{N}^*$, let $A^r = \{a_1, \ldots, a_r\}$, with $a_1 < a_2 < \cdots < a_r$, be a subset of distinct points of R such that

$$\sup_{p \in R} \min_{1 \leq i \leq r} |p - a_i| = \mathrm{O}\left(\frac{1}{r}\right), \quad \text{as } r \to +\infty. \tag{2}$$

Furthermore, we suppose that $a_i \in T_n$ for each $i = 1, \ldots, r$, that is, each point of A^r is a knot of the partition associated with a degree of freedom of the parametric cubic space V_N. In addition, we define the linear form in $H^3(R; \mathbb{R}^k)$ by

$$\boldsymbol{\Phi}_i(\mathbf{v}) = \mathbf{v}(a_i), \quad \forall i = 1, \ldots, r, \ \forall \mathbf{v} \in H^3(R; \mathbb{R}^k). \tag{3}$$

Let L^r be a Lagrangian operator defined from $H^3(R;\mathbb{R}^k)$ into $\mathcal{M}_{r\times k}$ by

$$L^r\mathbf{v} = \left(\mathbf{v}(a_i)^t\right)^t_{i=1,\dots,r}$$

and we suppose that

$$\operatorname{Ker} L^r \cap \mathbb{P}_2(R;\mathbb{R}^k) = \{0\}. \tag{4}$$

The goal is to find an approximating curve $\boldsymbol{\Upsilon}$ of $\boldsymbol{\Upsilon}_0$ defined by a function $\boldsymbol{\sigma}$ of V_N that interpolates the data points of A^r and minimizes some semi-norms of the order less than or equal to 3 in the space V_N.

Let $\beta^r = (\boldsymbol{\beta}_i)^t_{i=1,\dots,r}$ be a data matrix of $\mathcal{M}_{r\times k}$. Let us consider the set

$$K = \{\mathbf{v}\in V_N \mid L^r\mathbf{v} = \beta^r\}$$

and the linear space

$$K_0 = \{\mathbf{v}\in V_N \mid L^r\mathbf{v} = 0\}.$$

Finally, we suppose that

$$N \ge r. \tag{5}$$

In this situation, for each parameter vector $\boldsymbol{\tau} = (\tau_1,\tau_2)\in\mathbb{R}^2$, $\tau_i \ge 0$ for $i = 1, 2$, we consider the following minimization problem: find $\boldsymbol{\sigma}^{N,r}_{\boldsymbol{\tau}} \in K$ such that

$$\forall \mathbf{v}\in K,\quad J^r_{\boldsymbol{\tau}}(\boldsymbol{\sigma}^{N,r}_{\boldsymbol{\tau}}) \le J^r_{\boldsymbol{\tau}}(\mathbf{v}), \tag{6}$$

$J^r_{\boldsymbol{\tau}}$ being the functional defined by

$$J^r_{\boldsymbol{\tau}}(\mathbf{v}) = \sum_{j=1}^{2}\tau_j\,|\mathbf{v}|^2_j + |\mathbf{v}|^2_3,\quad \forall\mathbf{v}\in V_N.$$

Remark 1. We can observe that the functional $J^r_{\boldsymbol{\tau}}$ involves two terms. The first term, weighted by the vector $\boldsymbol{\tau}$, represents some different conditions, as for example fairness conditions (see [7]), curvature, minimal energy etc., while the second term measures the degree of smoothness of the given function. The minimization of the Sobolev semi-norm $|\cdot|_3$ is introduced in order to reduce, as much as possible, any unwanted oscillations. □

Definition 2. The solution of problem (6), if it exists, is called the parametric interpolating cubic spline in R, relative to A^r, L^r, β^r and $\boldsymbol{\tau}$.

Let us prove the existence and the uniqueness of the solution of the problem (6). Before doing so, we need the following result.

Lemma 3. *The application* $(((\cdot,\cdot))) : H^3(R;\mathbb{R}^k)\times H^3(R;\mathbb{R}^k)\to\mathbb{R}$ *defined by*

$$(((\mathbf{u},\mathbf{v}))) = \left\langle L^r\mathbf{u}, L^r\mathbf{v}\right\rangle_{r,k} + \sum_{j=1}^{2}\tau_j(\mathbf{u},\mathbf{v})_j + (\mathbf{u},\mathbf{v})_3, \tag{7}$$

is an inner product on $H^3(R;\mathbb{R}^k)$ *associated with the norm* $[[\mathbf{v}]] = (((\mathbf{v},\mathbf{v})))^{\frac{1}{2}}$, *and it is also equivalent to the usual Sobolev norm* $\|\cdot\|$.

Proof. The fact that $(((\cdot,\cdot)))$ is an inner product follows from (4) and from the properties of the inner semi-products $(\cdot,\cdot)_j$ for $j = 1, 2, 3$.

The equivalence of the norms $[[\cdot]]$ and $\|\cdot\|$ is deduced immediately from [8, Th. 6.3.12]. □

Theorem 4. *The problem (6) has a unique solution which is also the unique solution of the following variational problem:*

$$\begin{cases} \boldsymbol{\sigma}^{N,r}_{\boldsymbol{\tau}} \in K, \\ \forall\mathbf{v}\in K_0,\quad \displaystyle\sum_{j=1}^{2}\tau_j(\boldsymbol{\sigma}^{N,r}_{\boldsymbol{\tau}},\mathbf{v})_j + (\boldsymbol{\sigma}^{N,r}_{\boldsymbol{\tau}},\mathbf{v})_3 = 0. \end{cases} \tag{8}$$

Proof. The proof is based on the theorem of projection over a closed convex set, so let us first prove in Step 1 that K is a non-empty closed and convex subset of V_N, and we then apply such a theorem over K in Step 2.

Step 1. For any $a_i\in A^r$, let $\mathbf{w}_{a_i}$ be the basis cubic function of V_N that verifies

$$\begin{cases} \mathbf{w}_{a_i}(a_i) = 1, & \mathbf{w}_{a_i}(s) = 0,\ \forall s\in T_n\setminus\{a_i\} \\ w'_{a_i}(x_l) = 0, & \text{for } l = 0, n. \end{cases}$$

We define the function $\mathbf{g} = \sum_{j=1}^{r}\beta_j\mathbf{w}_{a_j}$. By construction, we have that $\mathbf{g}\in V_N$ and $\mathbf{g}(a_i) = \sum_{j=1}^{r}\beta_j\mathbf{w}_{a_j}(a_i) = \beta_i$ for all $i = 1,\dots,r$. Hence, $\mathbf{g}\in K$.

As the operator L^r is continuous and $\{\beta^r\}$ is a closed subset of $\mathbb{R}^{r,k}$, then $K = (L^r)^{-1}\{\beta^r\}$ is a closed subset of V_N. The set K is convex as a consequence of the linearity of the operator L^r.

Moreover, it is easy to check that K_0 is a vector space. As for all $\mathbf{u}, \mathbf{v} \in K$, $\mathbf{u} - \mathbf{v} \in K_0$ (and so $L^r(\mathbf{u} - \mathbf{v}) = 0$), we conclude that K is an affine variety of V_N associated with the vector space K_0.

Step 2. As for all $\mathbf{v} \in K$, $L^r\mathbf{v} = \beta^r$, problem (6) is equivalent to finding a function $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} \in K$ such that

$$\forall \mathbf{v} \in K, \quad [\![\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}]\!] \leq [\![\mathbf{v}]\!], \tag{9}$$

where $[\![\cdot]\!]$ is the norm defined in Lemma 3. By considering that $H^3(R;\mathbb{R}^k)$ is equipped with such a norm, it follows, by applying the theorem of projection over a closed convex set [9], that there exists a unique element $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} \in K$, which is the projection of 0 over K and, hence, the solution of (9). Such an element is also characterized by the relation

$$(((-\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{w} - \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}))) \leq 0, \quad \forall \mathbf{w} \in K,$$

which is equivalent to

$$\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} \in K, \quad (((-\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{v}))) \leq 0, \ \forall \mathbf{v} \in K_0.$$

So (8) follows, taking into account that K_0 is a vector subspace and that, for all $\mathbf{v} \in K_0$, $L^r\mathbf{v} = 0$. $\square$

Theorem 5. *There exists one and only one* $(rk+1)$*-uplet* $(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mu) \in V_N \times \mathcal{M}_{r\times k}$ *such that*

$$\begin{cases} \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} \in K, \\ \forall \mathbf{v} \in V_N, \quad \displaystyle\sum_{j=1}^{2} \tau_j(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{v})_j + (\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{v})_3 + \langle L^r\mathbf{v}, \mu\rangle_{r,k} = 0 \end{cases} \tag{10}$$

where $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ *is the solution of* (6)*.*

Proof. Let us consider the family of functions $\varphi^i \in S_3(T_n)$, $i = 1, \ldots, r$, verifying

$$\varphi^i(a_j) = \delta_{ij}, \quad \forall i, j = 1, \ldots, r.$$

The previous conditions make sense from (5).

Now, we consider for each $\mathbf{v} \in V_N$ the function $\mathbf{w} = \mathbf{v} - \sum_{i=1}^{r} \boldsymbol{\Phi}_i(v)\varphi^i$, where $\boldsymbol{\Phi}_i(\mathbf{v})$ are given in (3). It is evident to see that $\mathbf{w} \in K_0$. In fact, for each $a_p \in A^r$, one has

$$\mathbf{w}(a_p) = \mathbf{v}(a_p) - \sum_{i=1}^{r} \boldsymbol{\Phi}_i(\mathbf{v})\varphi^i(a_p) = \mathbf{v}(a_p) - \sum_{i=1}^{r} \boldsymbol{\Phi}_i(\mathbf{v})\delta_{ip} = \mathbf{v}(a_p) - \mathbf{v}(a_p) = \mathbf{0}.$$

So, as $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ is the solution of (6), then from Theorem 4 it follows that $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} \in K$ and

$$\sum_{j=1}^{2} \tau_j(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{w})_j + (\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{w})_3 = 0.$$

By developing, for all $\mathbf{v} \in V_N$ we obtain

$$\sum_{j=1}^{2} \tau_j \left(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{v} - \sum_{i=1}^{r} \boldsymbol{\Phi}_i(\mathbf{v})\varphi^i\right)_j + \left(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{v} - \sum_{i=1}^{r} \boldsymbol{\Phi}_i(\mathbf{v})\varphi^i\right)_3 = 0.$$

By linearity, for all $\mathbf{v} \in V_N$ we have

$$\sum_{j=1}^{2} \tau_j(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{v})_j - \sum_{j=1}^{2} \tau_j \left(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \sum_{i=1}^{r} \boldsymbol{\Phi}_i(\mathbf{v})\varphi^i\right)_j + (\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{v})_3 - \left(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \sum_{i=1}^{r} \boldsymbol{\Phi}_i(\mathbf{v})\varphi^i\right)_3 = 0. \tag{11}$$

If we put $\mu = -\sum_{j=1}^{k} \left(\sum_{l=1}^{2} \tau_l((\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r})_j, \varphi^i)_{l,1} + ((\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r})_j, \varphi^i)_{3,1}\right)_{i=1,\ldots,r}$ which belongs to $\mathcal{M}_{r\times k}$, then

$$-\sum_{j=1}^{2} \tau_j \left(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \sum_{i=1}^{r} \boldsymbol{\Phi}_i(\mathbf{v})\varphi^i\right)_j - \left(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \sum_{i=1}^{r} \boldsymbol{\Phi}_i(\mathbf{v})\varphi^i\right)_3 = \langle L^r\mathbf{v}, \mu\rangle_{r,k}$$

where $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} = \left((\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}{}_j)\right)_{j=1,\ldots,k}$. So, from (11), we obtain

$$\sum_{j=1}^{2} \tau_j(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{v})_j + (\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \mathbf{v})_3 + \langle L^r\mathbf{v}, \mu\rangle_{r,k} = 0, \quad \forall \mathbf{v} \in V_N.$$

Now let $(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r},\mu)$ be a solution of (10). It is clear that $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}\in K$. Moreover, for all $\mathbf{v}\in K_0$ the last expression becomes

$$\sum_{j=1}^{2}\tau_j(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r},\mathbf{v})_j+(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r},\mathbf{v})_3=0.$$

So, $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ is the solution of the problem (8), and it is therefore unique. Let $(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r},\overline{\mu})$ be another solution of (10). Then

$$\sum_{j=1}^{2}\tau_j(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r},\mathbf{v})_j+(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r},\mathbf{v})_3+\langle L^r\mathbf{v},\mu\rangle_{r,k}=0,\quad\forall\mathbf{v}\in V_N,$$

$$\sum_{j=1}^{2}\tau_j(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r},\mathbf{v})_j+(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r},\mathbf{v})_3+\langle L^r\mathbf{v},\overline{\mu}\rangle_{r,k}=0,\quad\forall\mathbf{v}\in V_N.$$

By subtracting we deduce that $\langle L^r\mathbf{v},\mu-\overline{\mu}\rangle_{r,k}=0$, and using (5), we conclude that $\mu=\overline{\mu}$ and finally, $(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r},\mu)$ is unique. $\square$

4. Computation

In this section, we show how to obtain and to compute in practice the solution of our problem. To this end, let $\{w_1,\dots,w_{n+3}\}$ be the basis functions of the space $S_3(T_n)$, that is, the functions that interpolate the knots of the partition T_n.

Therefore, $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ could be expressed as $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}=\sum_{i=1}^{n+3}\boldsymbol{\alpha}_i w_i$, where $\boldsymbol{\alpha}_i\in\mathbb{R}^k$ for $i=1,\dots,n+3$, which in principle are unknown. Applying the relation (10) for all $v\in S_3(T_n)$ we obtain

$$\sum_{i=1}^{n+3}\alpha_i^j\left(\sum_{l=1}^{2}\tau_l(w_i,v)_{l,1}+(w_i,v)_{3,1}\right)+\sum_{i=1}^{r}v(a_i)\mu_{ij}=0,\quad\forall j=1,\dots,k$$

subject to the restrictions

$$\sum_{l=1}^{k}\left(\sum_{i=1}^{n+3}\alpha_i^l w_i(a_j)\right)=\beta_j^l,\quad\text{for } j=1,\dots,r,$$

with $\beta=(\beta_i^l)_{\substack{j=1,\dots,r\\ l=1,\dots,k}}$ and $\boldsymbol{\alpha}_i=(\alpha_i^l)_{l=1,\dots,k}$ for all $i=1,\dots,r$. Taking $v=w_i$, for $i=1,\dots,n+3$, we obtain the linear system of the order $n+3+r$ with the unknown

$$\alpha_1^j,\dots,\alpha_{n+3}^j,\qquad \mu_{1j},\dots,\mu_{rj},\quad\text{for } j=1,\dots,k.$$

The matrix form of each system is

$$\begin{pmatrix}\mathcal{R} & D\\ D^t & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}^j\\ \boldsymbol{\mu}^j\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ \boldsymbol{\beta}_j^r\end{pmatrix},\quad\text{for } j=1,\dots,k,\tag{12}$$

where

$$\boldsymbol{\alpha}^j=\left(\alpha_i^j\right)_{1\le i\le n+3}^t,\qquad D=\left(w_i(a_j)\right)_{\substack{1\le j\le r\\ 1\le i\le n+3}},\qquad \boldsymbol{\beta}_j^r=\left(\beta_i^j\right)_{1\le i\le r}^t,\quad\text{for } j=1,\dots,k,$$

$$\mathcal{R}=\left((w_i,w_j)_{3,1}+\sum_{l=1}^{2}\tau_l(w_i,w_j)_{l,1}\right)_{1\le i,j\le n+3}=\mathcal{R}_0+\sum_{l=1}^{2}\tau_l\mathcal{R}_l,$$

with $\mathcal{R}_0=\left((w_i,w_j)_{3,1}\right)_{1\le i,j\le n+3}$, $\qquad\mathcal{R}_l=\left((w_i,w_j)_{l,1}\right)_{1\le i,j\le n+3}$, $\quad$ for $l=1,2$.

Finally, we can prove that the matrix $\mathcal{R}$ is symmetric positive definite.

Proposition 6. *The matrix $\mathcal{R}$ of the linear system given in (12) is symmetric positive definite and of the band type.*

Proof. It is obvious that $\mathcal{R}$ is symmetric. Now let $\mathbf{x}=(x_1,\dots,x_{n+3})\in\mathbb{R}^{n+3}$. Then, we have

$$\begin{aligned}\mathbf{x}\mathcal{R}\mathbf{x}^t&=\sum_{i,j=1}^{n+3}x_i\left(\sum_{l=1}^{2}\tau_l(w_i,w_j)_{l,1}+(w_i,w_j)_{3,1}\right)x_j\\&=\sum_{l=1}^{2}\tau_l\left(\sum_{i=1}^{n+3}x_iw_i,\sum_{j=1}^{n+3}x_jw_j\right)_{l,1}+\left(\sum_{i=1}^{n+3}x_iw_i,\sum_{j=1}^{n+3}x_jw_j\right)_{3,1}\\&=\sum_{l=1}^{2}\tau_l(w,w)_{l,1}+(w,w)_{3,1}\ge 0,\end{aligned}$$

with $w=\sum_{i=1}^{n+3}x_iw_i$. Furthermore, if $\mathbf{x}\mathcal{R}\mathbf{x}^t=0$, then taking into account Lemma 3 one has that $[\![w]\!]=0$, which implies $w=0$.

Hence, as the family $\{w_i\}_{1\le i\le n+3}$ is linearly independent, we deduce that $x_i = 0$, for each $i = 1, \dots, n+3$. Thus $\mathbf{x} = \mathbf{0}$, and $\mathcal{R}$ is a positive definite matrix.

Finally, the matrix $\mathcal{R}$ is of the band type because each function w_i has a reduced local support. □

5. Convergence

Under adequate hypotheses, we are going to prove that the parametric interpolating cubic spline in V_N relative to A^r, L^r, $L^r\mathbf{f}$ and $\boldsymbol{\tau}$ converges to $\mathbf{f}$ as r tends to $+\infty$. Before doing this, we need the following results, whose proofs adapting to the curve case, can be referred to in [7].

Corollary 7. *Suppose that the hypotheses (2) and (4) hold. Then, there exist $\nu > 0$ and, for all $r \in \mathbb{N}$, a subset A_0^r of A^r and a constant $C > 0$ such that, for all $r \ge \frac{C}{\nu}$, the application $[\![\cdot]\!]_0^r$ defined by*

$$[\![\mathbf{v}]\!]_0^r = \left(\sum_{a\in A_0^r} \langle \mathbf{v}(a)\rangle_k^2 + |\mathbf{v}|_3^2\right)^{1/2},$$

is a norm on $H^3(R;\mathbb{R}^k)$ uniformly equivalent, with respect to r, to the norm $\|\cdot\|$.

We remember that the Tchebycheff norm on $C^0(R;\mathbb{R}^k)$ is defined by

$$\|\mathbf{u}\|_T = \max_{p\in R} \langle \mathbf{u}(p)\rangle_k .$$

Let us give the easy relation between the semi-norms of order l and the Tchebycheff norm on $H^3(R;\mathbb{R}^k)$.

Lemma 8. *For all $\mathbf{u} \in C^2(R;\mathbb{R}^k)$ one has*

$$|\mathbf{u}|_\ell \le \sqrt{b-a}\,\|\mathbf{u}^{(\ell)}\|_T, \quad \ell = 0, 1, 2.$$

Now, let $\mathbf{f} \in C^4(R;\mathbb{R}^k)$ and let $\mathbf{s}_N$ be the parametric cubic spline of V_N solving the interpolation problem

$$\begin{cases} \mathbf{s}_N(x_i) = \mathbf{f}(x_i), & \text{for } 0 \le i \le n;\\ \mathbf{s}_N'(x_i) = \mathbf{f}'(x_i), & \text{for } i = 0, n.\end{cases}$$

So, from [10, Sec. 4.5] and applying Lemma 8 it follows that

$$|\mathbf{f} - \mathbf{s}_N|_\ell \le C_\ell h^{4-\ell}, \quad \ell = 0, 1, 2, 3, \tag{13}$$

where C_ℓ depends only on the Tchebycheff norms of derivatives of $\mathbf{f}$ less than ℓth, for $\ell = 0, 1, 2, 3$ and $h = \frac{b-a}{n}$.

Now, let $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ be the parametric interpolating cubic spline in V_N relative to A^r, L^r, $L^r\mathbf{f}$ and $\boldsymbol{\tau}$.

Theorem 9. *Suppose that the hypotheses (1) and (4) hold and that*

$$\tau_i = \mathrm{o}(1), \quad \text{as } r \to +\infty,\ \forall i = 1, 2. \tag{14}$$

Then, one has

$$\lim_{r\to+\infty} \left\|\mathbf{f} - \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}\right\| = 0.$$

Proof. First, as $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} - \mathbf{s}_N \in K_0$ and

$$K_0 \subset K_0' = \{\mathbf{v} \in H^3(R;\mathbb{R}^k) | L^r\mathbf{v} = 0\},$$

we deduce, from Theorem 5 of [2], that

$$\sum_{l=1}^{2} \tau_l(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}, \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} - \mathbf{s}_N)_l + \left(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}, \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} - \mathbf{s}_N\right)_3 = 0, \tag{15}$$

where $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}$ is the parametric interpolating variational spline in $H^3(R;\mathbb{R}^k)$ that interpolates the set A^r (see [2, Def. 3]).

Likewise, from Theorem 4, we have

$$\sum_{l=1}^{2} \tau_l(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} - \mathbf{s}_N)_l + \left(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}, \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} - \mathbf{s}_N\right)_3 = 0. \tag{16}$$

Hence, from (15) and (16) one has

$$\sum_{l=1}^{2} \tau_l(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} - \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}, \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} - \mathbf{s}_N)_l + \left(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} - \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}, \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} - \mathbf{s}_N\right)_3 = 0,$$

which means

$$\sum_{l=1}^{2}\tau_l\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right|_l^2+\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right|_3^2=\sum_{l=1}^{2}\tau_l(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-\mathbf{s}_N,\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N)_l+\left(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-\mathbf{s}_N,\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right)_3.$$

It follows that

$$\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right|_3^2\le\sum_{l=1}^{2}\tau_l\left(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-\mathbf{s}_N,\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right)_l+\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-\mathbf{s}_N\right|_3\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right|_3.$$

From (14) we deduce that

$$\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right|_3^2\le o(1)\sum_{l=1}^{2}\left(\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-\mathbf{s}_N\right|_l+\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right|_l\right)+\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-\mathbf{s}_N\right|_3\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right|_3,\ r\to+\infty. \tag{17}$$

Furthermore, we have for $l=1,2$ that

$$\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-\mathbf{s}_N\right|_l\le\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-\mathbf{s}_N\|,$$
$$\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right|_l\le\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\|.$$

Then, from (17), we have that there exist $C>0$ and $r_0\in\mathbb{N}$ such that for all $r\ge r_0$,

$$\left|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right|_3^2\le C\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-S_N\|\,\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\|.$$

As we have $L^r(\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N)=0$, we deduce that there exist $C>0$ and $r_0\in\mathbb{N}$ such that for all $r\ge r_0$,

$$\left\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right\|^2\le C\left\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-\mathbf{s}_N\right\|\left\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right\|,$$

which means that

$$\left\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right\|\le C\left\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-\mathbf{s}_N\right\|. \tag{18}$$

Now, it is verified that

$$\left\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{f}\right\|\le\left\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{s}_N\right\|+\|\mathbf{s}_N-\mathbf{f}\|,$$

and, from (18), we have that there exist $C>0$ and $r_0\in\mathbb{N}$ such that for all $r\ge r_0$,

$$\left\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{f}\right\|\le C\left\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{r}-\mathbf{s}_N\right\|+\|\mathbf{s}_N-\mathbf{f}\|,$$

which implies, together with (13) and [2, Theorem 9], that

$$\lim_{r\to+\infty}\left\|\mathbf{f}-\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}\right\|=0.$$

6. Optimization of the parameters

In this section, we study a method to optimize the parameter values of $\boldsymbol{\tau}$ by an approximation method. For a fixed r, this method consists of minimizing the relative error $\|\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}-\mathbf{f}\|_0^2$. To this end, let $\{w_1,\dots,w_{n+3}\}$ be basic functions of $S_3(T_n)$ with local support. We remember that $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}=\sum_{i=1}^{n+3}\boldsymbol{\alpha}_i w_i$ with $\boldsymbol{\alpha}_i\in\mathbb{R}^k$, $i=1,\dots,n+3$. Hence, in order to obtain an optimal vector, we look for the matrix $\alpha_0=(\boldsymbol{\alpha}_{0i})_{1\le i\le n+3}\in\mathcal{M}_{n+3\times k}$ as the solution of the following minimization problem.

Find $(\alpha_0,\mu_0)^t\in\mathcal{M}_{(n+3+r)\times k}$ such that $F(\alpha_0,\mu_0)=\min_{(\alpha,\mu)^t\in\mathcal{M}_{(n+3+r)\times k}}F(\alpha,\mu)$, where

$$F(\alpha,\mu)=\left\|\sum_{i=1}^{n+3}\boldsymbol{\alpha}_i w_i-\mathbf{f}\right\|_0^2+\langle\mu,D^t\alpha-\beta^r\rangle_{r,k}^2$$

and $\beta^r=(\mathbf{f}(a_i))_{i=1,\dots,r}^t\in\mathcal{M}_{r\times k}$. Since, for any $l=1,\dots,k$, we have

$$\begin{cases}\dfrac{\partial F}{\partial\alpha_j^l}=2\left(\displaystyle\sum_{i=1}^{n+3}\alpha_i^l w_i-f_l,w_j\right)_{0,1}+\displaystyle\sum_{i=1}^{r}\mu_i^l w_j(a_i), & \forall j=1,\dots,n+3,\\ \dfrac{\partial F}{\partial\mu_j^l}=\displaystyle\sum_{i=1}^{n+3}\alpha_j^l w_j(a_i)-f_l(a_i), & \forall j=1,\dots,r,\end{cases}$$

with $\mathbf{f}=(f_1,\dots,f_k)\in C^4(R;\mathbb{R}^k)$.

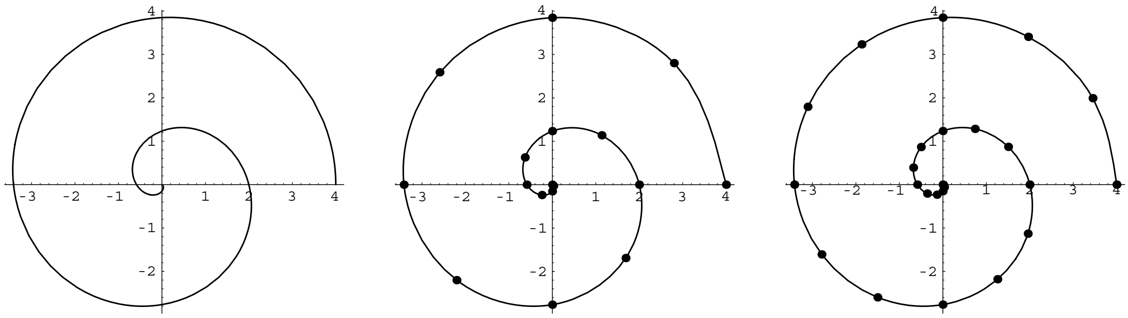

Fig. 1. From left to right, the graph of the original curve Υ_0. The graph of an optimal curve parameterized by the parametric interpolating spline $\sigma_{\tau}^{N,r}$ from $r = 17$ interpolation points, with $n = 32$ uniform partition of the interval [0, 1]. The value of the optimal parameter $\tau = \{1.63701 \times 10^{-5}, 2.94751 \times 10^{-5}\}$, $N = k(n+3) = 2(32+3) = 70$. And the graph of another optimal curve parameterized by the parametric interpolating spline $\sigma_{\tau}^{N,r}$ from $r = 25$ interpolation points, with $n = 48$ uniform partition of the interval [0, 1], the value of the optimal parameter $\tau = \{3.0621 \times 10^{-5}, 5.41768 \times 10^{-5}\}$, $N = k(n+3) = 2(48+3) = 102$.

In general, there would not exist values of the parameter $\tau = (\tau_1, \tau_2)$ such that $(\alpha_0, \mu_0) = (\alpha, \mu)$ is the solution of (12), which we can rewrite as

$$\begin{cases} \mathcal{R}\alpha + D\mu = 0, \\ D^t\alpha = \beta^r. \end{cases}$$

If we call

$$B'_l = \begin{pmatrix} \mathcal{R}_l & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \mu_0 \end{pmatrix}, \quad l = 1, 2, \qquad R'_1 = \begin{pmatrix} \mathcal{R}_0 & D \\ D^t & 0 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \mu_0 \end{pmatrix}, \qquad \widehat{b} = \begin{pmatrix} 0 \\ \beta^r \end{pmatrix},$$

it means that we could not find $\tau \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $\tau_1 B'_1 + \tau_2 B'_2 = \widehat{b} - R'_1 = B'$. We will then consider the best approximation of $\widehat{b} - R'_1$ in the linear space

$$\overline{\xi} = \left\{ \sum_{l=1}^{2} \tau_l B'_l \in \mathcal{M}_{(n+3+r)\times k},\ \tau_i \geq 0,\ i = 1, 2 \right\},$$

that is, we look for the solution of the optimization problem:

$$\min_{\text{s.t.}\,\tau_l \geq 0, l=1,2} \left\langle \widehat{b} - R'_1 - \sum_{l=1}^{2} \tau_l B'_l \right\rangle^2_{n+3+r,k}. \tag{19}$$

Obviously, if there exists a parameter vector $\tau^* = (\tau_1^*, \tau_2^*)$ such that the solution of the linear system (12) is α_0, then τ^* is the solution to problem (19). Since $\overline{\xi}$ is a convex closed subset of $\mathcal{M}_{(n+3+r)\times k}$, problem (19) has a unique solution $\sum_{l=1}^{2} \tau_l^* B'_l$. Moreover, by applying Kuhn–Tucker conditions (see e.g. [11]) we find that if $\tau^* = (\tau_1^*, \tau_2^*)$ is the solution to problem (19), meaning there exists $\{\lambda_1, \lambda_2\}$ such that

$$\begin{cases} -2\left\langle B'_i, B' - \displaystyle\sum_{l=1}^{2} \tau_l^* B'_l \right\rangle_{n+3+r,k} - \lambda_i = 0, & \forall i = 1, 2, \\ \lambda_i \tau_i^* = 0, & \forall i = 1, 2, \\ \lambda_i \geq 0, & \forall i = 1, 2, \\ \tau_i^* \geq 0, & \forall i = 1, 2. \end{cases} \tag{20}$$

To compute all the solutions of problem (20) we consider, for any subset s of $\{1, 2\}$, the couple $\{\tau_{s,j}\}_{j=1,2}$ and $\{\lambda_{s,j}\}_{j=1,2}$ in which $\lambda_{s,j} = 0$ if, and only if, $j \in s$. In order that $\{\tau_{s,j}\}_{j=1,2}$ and $\{\lambda_{s,j}\}_{j=1,2}$ be the solution of (20), $\tau_{s,j} = 0$ must hold if $j \notin s$. The elements $\tau_{s,j}$, for $j \in s$, can be obtained as the solutions of the linear system $\left\langle B'_i, B' - \sum_{l=1}^{2} \tau_l^* B'_l \right\rangle_{n+3+r,k} = 0$ for $i \in s$, and the elements $\lambda_{s,j}$ for j not belonging to s, can be obtained as $\lambda_{s,j} = -2\left\langle B'_i, B' - \sum_{l=1}^{2} \tau_l^* B'_l \right\rangle_{n+3+r,k}$.

The values $\{\tau_{s,j}\}_{j=1,2}$ and $\{\lambda_{s,j}\}_{j=1,2}$ obtained in this way are solutions to problem (20), if they satisfy $\tau_{s,j} \geq 0$ and $\lambda_{s,j} \geq 0$ for all $j \in \{1, 2\}$.

7. Numerical and graphical examples

Now, in order to test our methodology, let us present some examples that confirm our theoretical results. To this end, for each integer r let A^r contain r arbitrary knots of the uniform or not uniform partition of $\overline{R}$.

Table 1
Table of some values of estimation of error E_r from a uniform partition of the interval [0, 1] into n equal intervals, r interpolation points and distinct values of $\{\tau_1, \tau_2\}$. The optimal value of $\boldsymbol{\tau}$ is designated by $\{\tau_1, \tau_2\}^*$.

n	r	$\{\tau_1, \tau_2\}$	$E_r = \|\mathbf{f} - \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}\|_0^2$
32	17	$\{1.63701 \times 10^{-5}, 2.94751 \times 10^{-5}\}^*$	8.44856×10^{-4}
		$\{10^{-4}, 1\}$	2.12671×10^{-2}
		$\{1, 10^{-5}\}$	1.21717×10^{-3}
		$\{10^{-5}, 10^{-6}\}$	8.44858×10^{-4}
48	25	$\{3.0621 \times 10^{-5}, 5.41768 \times 10^{-5}\}^*$	2.53607×10^{-4}
		$\{10^{-6}, 1\}$	2.56083×10^{-4}
		$\{1, 10^{-5}\}$	3.10453×10^{-4}
		$\{10^{-3}, 10^{-6}\}$	7.43246×10^{-3}

By using the numerical Cholesky method, we have solved the linear system given in (12) to compute our spline $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ for different values of data r, $\boldsymbol{\tau}$ and N. Meanwhile, using the method developed in Section 6 we have calculated different values of the optimal parameter $\boldsymbol{\tau}$.

Likewise, we have computed the estimation of error $E_r = \|\mathbf{f} - \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}\|_0^2$ applying the Simpson formula composed by a partition of R in intervals.

7.1. Uniform partition

Let us begin with a uniform partition. We consider the planar curve $\boldsymbol{\Upsilon}_0$ defined by

$$\mathbf{f}(x) = \{2\cos(4\pi x)(1 + \cos(\pi x)), 2\sin(4\pi x)(1 + \cos(\pi x))\}$$

as a test function, whose graph appearing in Fig. 1 (left side).

For $r = 7$ interpolation points and $n = 12$ uniform partition, the expression of $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} = \left((\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r})_1, (\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r})_2\right)$ is

$$\begin{cases}
(4 - 45.8996x + 41.9878x^2 + 124.409x^3, 53.9167x - 148.11x^2 - 384.676x^3), & 0 \le x \le \frac{1}{12}, \\
(3.95839 - 44.4016x + 24.0123x^2 + 196.311x^3, -0.141025 + 58.9936x - 209.033x^2 - 140.985x^3), & \frac{1}{12} \le x \le \frac{2}{12}, \\
(4.39404 - 52.2433x + 71.0622x^2 + 102.211x^3, -6.55584 + 174.46x - 901.834x^2 + 1244.62x^3), & \frac{2}{12} \le x \le \frac{3}{12}, \\
(8.74134 - 104.411x + 279.733x^2 - 176.016x^3, -4.15801 + 145.686x - 786.738x^2 + 1091.15x^3), & \frac{3}{12} \le x \le \frac{4}{12}, \\
(29.5856 - 292.009x + 842.528x^2 - 738.812x^3, 61.5407 - 445.602x + 987.128x^2 - 682.712x^3), & \frac{4}{12} \le x \le \frac{5}{12}, \\
(12.2745 - 167.369x + 543.392x^2 - 499.502x^3, 84.3758 - 610.015x + 1381.72x^2 - 998.384x^3), & \frac{5}{12} \le x \le \frac{6}{12}, \\
(-129.555 + 683.61x - 1158.57x^2 + 635.137x^3, -42.1111 + 148.906x - 136.124x^2 + 13.5112x^3), & \frac{6}{12} \le x \le \frac{7}{12}, \\
(-145.102 + 763.567x - 1295.64x^2 + 713.461x^3, -111.549 + 506.016x - 748.313x^2 + 363.333x^3), & \frac{7}{12} \le x \le \frac{8}{12}, \\
(132.918 - 487.524x + 581.001x^2 - 224.857x^3, -54.5437 + 249.492x - 363.526x^2 + 170.94x^3), & \frac{8}{12} \le x \le \frac{9}{12}, \\
(226.953 - 863.664x + 1082.52x^2 - 447.754x^3, -1.1259 + 35.8208x - 78.6311x^2 + 44.32x^3), & \frac{9}{12} \le x \le \frac{10}{12}, \\
(-8.07884 - 17.5507x + 67.1848x^2 - 41.62x^3, 46.979 - 137.357x + 129.182x^2 - 38.8053x^3), & \frac{10}{12} \le x \le \frac{11}{12}, \\
(-94.2669 + 264.519x - 240.528x^2 + 70.2756x^3, 45.5435 - 132.659x + 124.057x^2 - 36.9416x^3), & \frac{11}{12} \le x \le 1.
\end{cases}$$

7.2. Not uniform partition

Now, we take a not uniform partition to show that our method is also valid for this kind of partition. We consider the new curve $\boldsymbol{\Upsilon}_0$ defined by

$$\mathbf{f}(x) = \{2\cos(2\pi x)(1 + \cos(2\pi x)), 2\sin(2\pi x)(1 + \cos(2\pi x))\}$$

as a test function, its graph appearing in Fig. 2 (left side).

Table 2
Table of some values of estimation of error E_r from a not uniform partition of the interval [0, 1] into n random intervals, r interpolation points and distinct values of $\{\tau_1, \tau_2\}$. The optimal value of $\boldsymbol{\tau}$ is designated by $\{\tau_1, \tau_2\}^*$.

n	r	$\{\tau_1, \tau_2\}$	$E_r = \Vert \mathbf{f} - \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} \Vert_0^2$
25	15	$\{1.1358 \times 10^{-5}, 2.06996 \times 10^{-5}\}^*$	6.12991×10^{-2}
		$\{10^{-3}, 10^{-4}\}$	7.83645×10^{-2}
		$\{10^{-5}, 10^{-4}\}$	6.12998×10^{-2}
		$\{10^{-8}, 10^{-8}\}$	6.13462×10^{-2}
48	35	$\{5.68348 \times 10^{-6}, 1.08529 \times 10^{-5}\}^*$	1.00441×10^{-3}
		$\{10^{-3}, 10^{-2}\}$	1.1164×10^{-3}
		$\{0, 0\}$	9.64232×10^{-3}
		$\{10^{-7}, 10^{-9}\}$	1.100639×10^{-3}

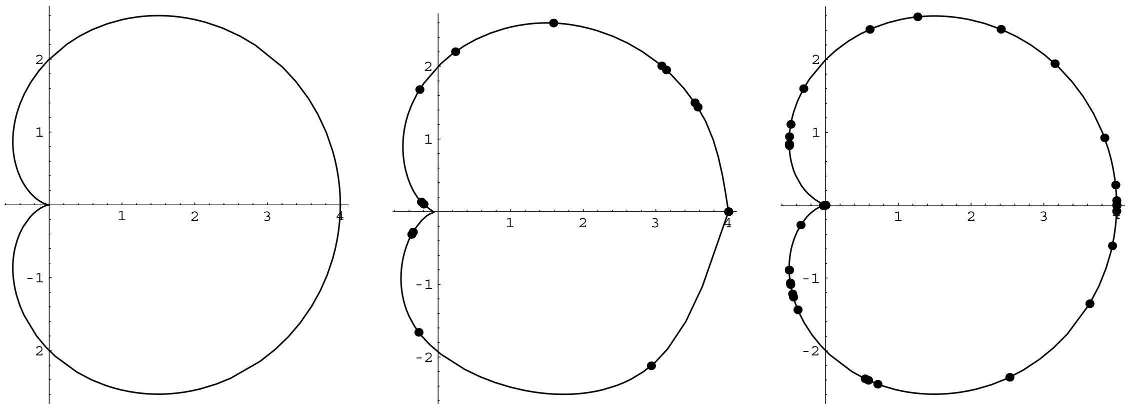

Fig. 2. From left to right, the graph of the original curve $\boldsymbol{\Upsilon}_0$. The graph of an optimal curve parameterized by the parametric interpolating spline $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ from $r = 15$ random interpolating points, with $n = 25$ not uniform partition of [0, 1]. The value of the optimal parameter $\boldsymbol{\tau} = \{1.1358 \times 10^{-5}, 2.06996 \times 10^{-5}\}$, $N = k(n+3) = 2(25+3) = 56$. And the graph of another optimal curve parameterized by the parametric interpolating spline $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ from $r = 25$ random interpolation points, with $n = 48$ not uniform partitions of [0, 1], the value of the optimal parameter $\boldsymbol{\tau} = \{5.68348 \times 10^{-6}, 1.08529 \times 10^{-5}\}$, $N = k(n+3) = 2(48+3) = 102$.

For $r = 6$ interpolation points and $n = 10$ not uniform partitions, the expression of $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r} = \left((\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r})_1, (\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r})_2\right)$ is

$$\begin{cases}
(4 - 16.3724x + 15.5602x^2 + 15.8586x^3, 29.653x - 73.0538x^2 - 72.325x^3), & 0 \le x \le \frac{1}{12}, \\
(4.00115 - 16.4136x + 16.0552x^2 + 13.879x^3, -0.0115452 + 30.0686x - 78.0413x^2 - 52.3749x^3), & \frac{1}{12} \le x \le \frac{1}{7}, \\
(4.13215 - 19.1648x + 35.3132x^2 - 31.0566x^3, -0.732408 + 45.2067x - 184.008x^2 + 194.881x^3), & \frac{1}{7} \le x \le \frac{8}{23}, \\
(3.04768 - 9.8112x + 8.42163x^2 - 5.28545x^3, -4.4564 + 77.3261x - 276.351x^2 + 283.377x^3), & \frac{8}{23} \le x \le \frac{4}{9}, \\
(-3.52542 + 34.5572x - 91.4074x^2 + 69.5863x^3, 22.6823 - 105.86x + 135.818x^2 - 25.7505x^3), & \frac{4}{9} \le x \le \frac{8}{13}, \\
(5.81315 - 10.9683x - 17.4284x^2 + 29.5143x^3, -26.086 + 131.885x - 250.518x^2 + 183.515x^3), & \frac{8}{13} \le x \le \frac{2}{3}, \\
(28.7967 - 114.395x + 137.711x^2 - 48.0553x^3, -128.944 + 594.748x - 944.813x^2 + 530.662x^3), & \frac{2}{3} \le x \le \frac{3}{4}, \\
(19.3007 - 76.4102x + 87.0652x^2 - 25.5461x^3, -130.038 + 599.123x - 950.646x^2 + 533.255x^3), & \frac{3}{4} \le x \le \frac{4}{5}, \\
(10.5332 - 43.5321x + 45.9676x^2 - 8.42208x^3, -26.6294 + 211.34x - 465.917x^2 + 331.285x^3), & \frac{4}{5} \le x \le \frac{5}{6}, \\
(-9.0716 + 27.0451x - 38.7251x^2 + 25.455x^3, 22.8204 - 29.7276x - 74.1822x^2 + 119.095x^3), & \frac{5}{6} \le x \le 1.
\end{cases}$$

7.3. Comparison with other methods

Let us now compare our method with some classical ones, such as the interpolating natural cubic spline [10] and the interpolating D^m splines [1].

Table 3
Table of some values of estimation of error E_r from some uniform partitions of the interval [0, 1] into n equal intervals, r interpolation points for both interpolating natural cubic splines, and D^m splines.

Method	n	r	$E_r = \|\mathbf{f} - \boldsymbol{\sigma}\|_0^2$
Natural spline	8	9	3.94644×10^{-1}
	32	17	6.14451×10^{-3}
	48	25	3.27154×10^{-3}
D^mspline	16	9	2.58138×10^{-1}
	32	17	1.2543×10^{-3}
	48	25	8.7926×10^{-4}

Table 4
Table of some values of estimation of error E_r from some not uniform partitions of the interval [0, 1] into n intervals, r interpolation points for both interpolating natural cubic splines, and D^m splines.

Method	n	r	$E_r = \|\mathbf{f} - \boldsymbol{\sigma}\|_0^2$
Natural spline	8	9	1.13434
	25	15	4.17239×10^{-1}
	48	35	1.77567×10^{-1}
D^mspline	16	9	6.25777
	25	15	1.70698×10^{-1}
	45	35	1.184837×10^{-2}

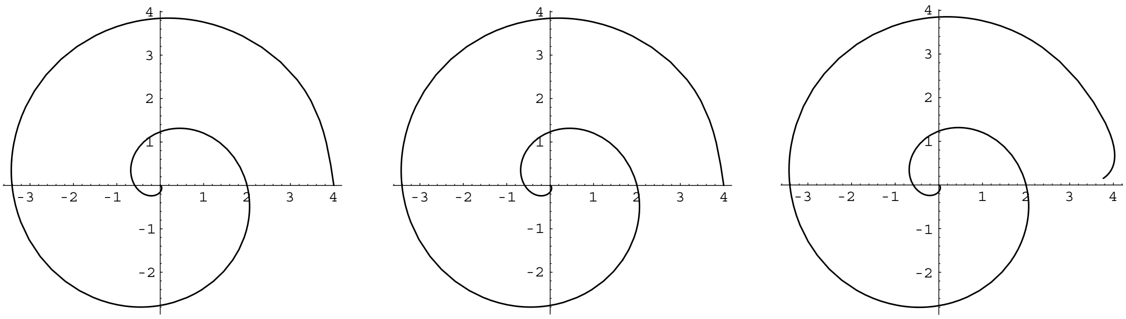

Fig. 3. From left to right, the graph of an approximating optimal curve defined by $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$, the graph of an approximating curve defined by D^m-spline and the graph of an approximating curve defined by interpolating natural cubic spline, with $r = 25$ interpolating points and $n = 48$ uniform partitions.

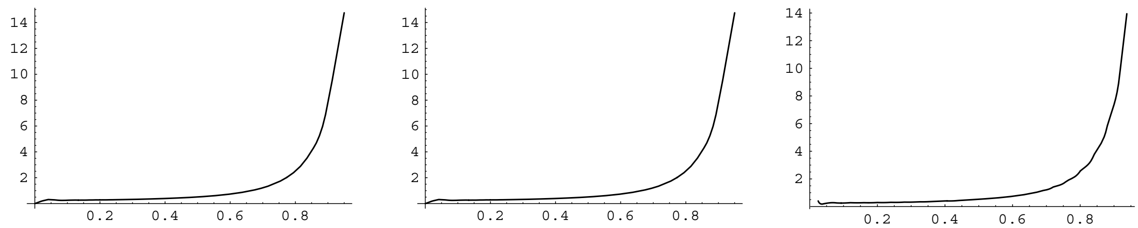

Fig. 4. From left to right, the shape of the curvature of an approximating curve defined by $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$, the shape of the curvature of an approximating curve defined by D^m-spline and the shape of the curvature of an approximating curve defined by interpolating natural cubic spline.

First, for the uniform partition we consider the same curve as used for the uniform partition in Section 7.1. In order that the comparison makes sense, we have used almost the same data, i.e. the same points of interpolation and the same partitions; accordingly, we can observe from Table 1 that the values of the relative error for our method are better with respect to those of the methods presented in Table 3.

Second, for a not uniform partition, the same analysis was done for some not uniform partitions. Tables 2 and 4, lead us to the same conclusion, that this method is better than traditional ones.

In closing, Figs. 3–5 show some approximating curves and shapes of the curvatures defined by the aforementioned methods, with $r = 25$ interpolating points and $n = 48$ uniform partitions. We can do the same for a not uniform partition.

7.4. Approximation of some non-convex curves

This subsection shows that the presented theory is also valid for non-convex curves. Although the two above defined curves are not convex, we selected for this example a non-convex curve with clear points of inflection.

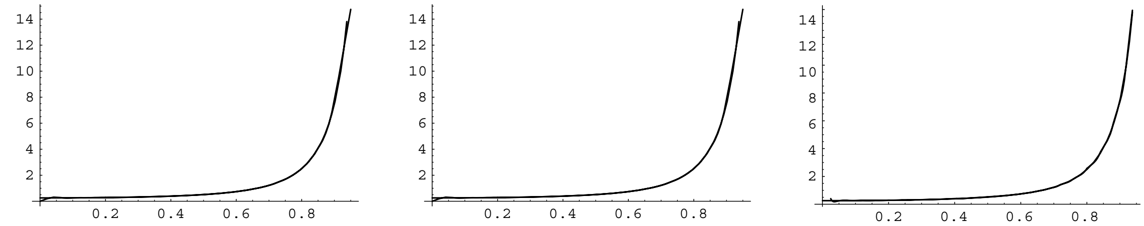

Fig. 5. From left to right, the shape of the curvature of the original curve together with the shape of the curvature of an approximating curve defined by, respectively, $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$, D^m-spline and interpolating natural cubic spline.

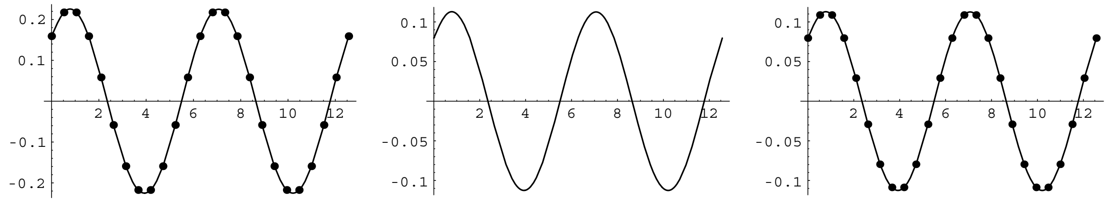

Fig. 6. From left to right, the graph of the original curve Υ_0 together with $r = 25$ interpolation points, the graph of an optimal curve parameterized by the parametric interpolating spline $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ and the graph of such spline $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ together with the $r = 25$ interpolation points; here $n = 48$ uniform partitions of the interval [0, 1], the value of the optimal parameter $\boldsymbol{\tau} = \{1.00905 \times 10^{-5}, 1.86358 \times 10^{-5}\}$. In this case, $E_r = 1.42592 \times 10^{-4}$.

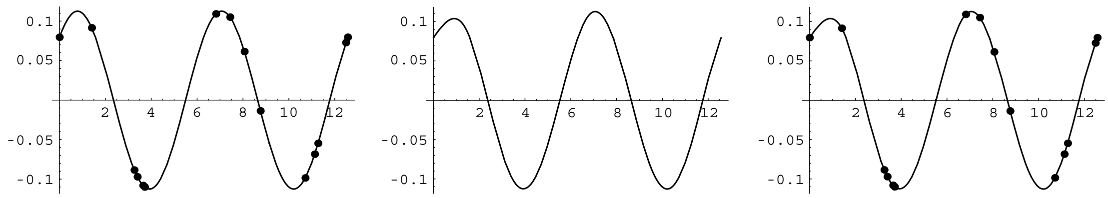

Fig. 7. From left to right, the graph of the original curve Υ_0 with $r = 15$ points, the graph of an optimal curve parameterized by the parametric interpolating spline $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ and the graph of $\boldsymbol{\sigma}_{\boldsymbol{\tau}}^{N,r}$ with $r = 15$ interpolation points; here $n = 25$ not uniform partitions of the interval [0, 1], the value of the optimal parameter $\boldsymbol{\tau} = \{2.80218 \times 10^{-4}, 0\}$. In this case, $E_r = 1.89415 \times 10^{-4}$.

We consider the new curve Υ_0 defined by the simple function

$$\mathbf{f}(x) = \left\{4\pi x, \frac{\sin(4\pi x) + \cos(4\pi x)}{4\pi}\right\}$$

as a test function, its graph appearing in Fig. 6 (left side).

Conclusion:

The graphs of the original curves given in Figs. 1 and 2, left sides, and those defined by our parametric interpolating cubic splines given in the same figures (right side) are similar. Likewise, in Tables 1 and 2 one can see that the numerical results are compatible with the theory presented in this work, especially the theorem of convergence. If we moreover take into account Sections 7.3 and 7.4, we can conclude that the methodology put forth in this paper serves well as an approximation (interpolation) approach (Fig. 7).

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